Sequential Games and Normal Form

Chess strategy: function from game states to moves

chess: ~10^{47} states branching factor ≈ 35

35 choices per state on average

\# of strategies $\underline{\ \ }_0\ \underline{\ \ }_1\ \underline{\ \ } \dots \underline{\ \ }_{10^{47}}$

$\approx 35^{(10^{47})}$ strategies

Chess as a normal form (matrix) game:

P1 picks strategy] simultaneous
P2 picks strategy

play game according to strategies

Columns: 0 ... $35^{(10^{47})}$

Rows: 0 ... $35^{(10^{47})}$

-1 black

0 draw

1 white

Value of a Game

Suppose there are two saddle points in mixed strategies (X_1^*, Y_1^*) (X_2^*, Y_2^*)

PI changing from equilibrium

$$E(X_1^*, Y_2^*) \le E(X_2^*, Y_2^*)$$
$$\le E(X_2^*, Y_1^*)$$
$$\le E(X_1^*, Y_1^*)$$
$$\le E(X_1^*, Y_2^*)$$

P2 changing from equilibrium

Squeeze - all $\le$ are $=$

$$E(X_2^*, Y_2^*) = E(X_1^*, Y_2^*)$$

call the value of the equilibria $v(A)$

Best response: Best response to $\overset{PI}{X}$ is strategy $\overset{P2}{Y}$ that minimizes $E(X,Y)$
$\underset{PII}{Y}$ is strategy $\underset{PI}{X}$ that maximizes $E(X,Y)$

Equilibrium strategies X^*, Y^* are best responses to each other

Finding Saddle Points in Mixed Strategies

Thm: X^*, Y^* is a saddle point in mixed strategies and value$(A) = E(X^*, Y^*)$

if and only if

$E(i, Y^*) \le E(X^*, Y^*) \le E(X^*, j)$ for all i, j

$\Rightarrow$: def of equilibrium

$\Leftarrow$: (want $E(X, Y^*) \le E(X^*, Y^*) \le E(X^*, Y)$ for all X, Y)

let X, Y be any strategies. Write $X = (x_1\ x_2 \cdots x_n)$ $Y = (y_1 \cdots y_m)$

$x_1 E(1, Y^*) \le x_1 E(X^*, Y^*)$

$x_2 E(2, Y^*) \le x_2 E(X^*, Y^*)$

$\vdots$

$+\ x_n E(n, Y^*) \le x_n E(X^*, Y^*)$

$\sum_{i=1}^{n} P(\text{play row } i) \cdot E(i, Y^*)$, $P(\text{play row } i) = x_i$

$E(X, Y^*) \le \underbrace{(x_1 + \cdots + x_n)}_{=1} \cdot E(X^*, Y^*) = E(X^*, Y^*) \le E(X^*, Y)$

works for P1 — works for P2

	F	C	S
F	0.30	0.25	0.20
C	0.26	0.33	0.28
S	0.28	0.30	0.33

Is $X^* = (\frac{2}{7}\ 0\ \frac{5}{7})$ $Y^* = (\frac{5}{7}\ \frac{2}{7}\ 0)$ an equilibrium?

$E(X^*, Y^*) = \frac{2}{7} \cdot \frac{5}{7} \cdot 0.30 + \frac{2}{7} \cdot \frac{2}{7} \cdot 0.25 + \frac{5}{7} \cdot \frac{5}{7} \cdot 0.28 + \frac{5}{7} \cdot \frac{2}{7} \cdot 0.30$

$= \frac{2}{7}$

$E(1, Y^*) = \frac{5}{7} \cdot 0.30 + \frac{2}{7} \cdot 0.25 = \frac{2}{7} \stackrel{?}{\le} \frac{2}{7}$ ✓

$E(2, Y^*) = \frac{5}{7} \cdot 0.26 + \frac{2}{7} \cdot 0.33 = \frac{196}{700} \stackrel{?}{\le} \frac{2}{7}$ ✓

$E(3, Y^*) = \frac{5}{7} \cdot 0.28 + \frac{2}{7} \cdot 0.30 = \frac{2}{7} \stackrel{?}{\le} \frac{2}{7}$

$E(X^*, 1) = \frac{2}{7} \cdot 0.30 + \frac{5}{7} \cdot 0.28 = \frac{2}{7} \stackrel{?}{\ge} \frac{2}{7}$ ✓

$E(X^*, 2) = \quad \stackrel{?}{\ge} \frac{2}{7}$ ✓

$E(X^*, 3) = \quad \stackrel{?}{\ge} \frac{2}{7}$ ✓

all inequalities true so (X^*, Y^*) is equilibrium

$X = (\frac{1}{2}\ \frac{1}{2}\ 0)$ $Y = (\frac{1}{2}\ \frac{1}{2}\ 0)$

$E(X, Y) = \frac{1}{2} \cdot \frac{1}{2} \cdot 0.3 + \frac{1}{2} \cdot \frac{1}{2} \cdot 0.25 + \frac{1}{2} \cdot \frac{1}{2} \cdot 0.26 + \frac{1}{2} \cdot \frac{1}{2} \cdot 0.33 = 0.285$

$E(1, Y) = 0.275 \le 0.285$

$\longrightarrow E(2, Y) = \frac{1}{2} \cdot 0.26 + \frac{1}{2} \cdot 0.33 = 0.295 > 0.285$ Not equilibrium!

$E(3, Y) = 0.29$

$\hookrightarrow$ 2 $(0\ 1\ 0)$ is best response to $Y = (\frac{1}{2}\ \frac{1}{2}\ 0)$

Can use to find solutions

	L	R
L	$\frac{1}{2}$	1
R	1	$\frac{2}{3}$

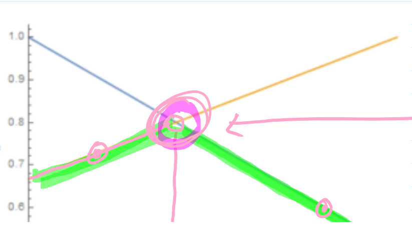
$X = (x\ \ 1-x)$

$E(X, 1) = x \cdot \frac{1}{2} + (1-x) \cdot 1 = 1 - \frac{1}{2}x$

$E(X, 2) = x \cdot 1 + (1-x) \cdot \frac{2}{3} = \frac{2}{3} + \frac{1}{3}x$

$Y = (y\ \ 1-y)$ $E(1, Y) = y \cdot \frac{1}{2} + (1-y) 1 = 1 - \frac{1}{2}y$

$E(2, Y) = y \cdot 1 + (1-y) \cdot \frac{2}{3} = \frac{2}{3} + \frac{1}{3}y$



$E(X, 2)$

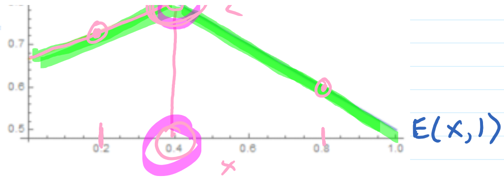
find intersection $1 - \frac{1}{2}x = \frac{2}{3} + \frac{1}{3}x$

$\frac{1}{3} = \frac{5}{6}x$

$\frac{2}{5} \quad \frac{6^2}{5} \cdot \frac{1}{3} = x$

$Y = (\frac{2}{5}\ \ \frac{3}{5})$

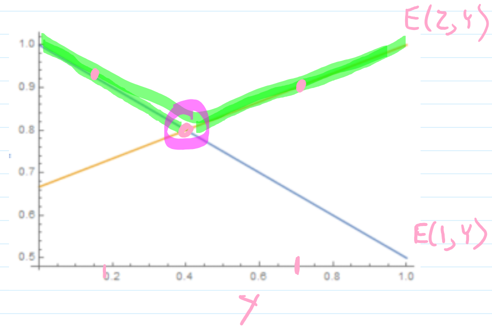
$E(?, Y)$



$\frac{1}{3} = \frac{5}{6}x$

$\frac{2}{5} = \frac{6}{5}\cdot\frac{1}{3} = x$

$y = \left(\frac{2}{5} \quad \frac{3}{5}\right)$

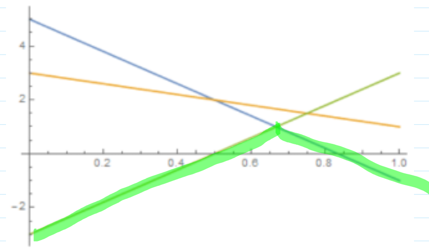


$A = \begin{pmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \end{pmatrix}$

$E(x,1) =$

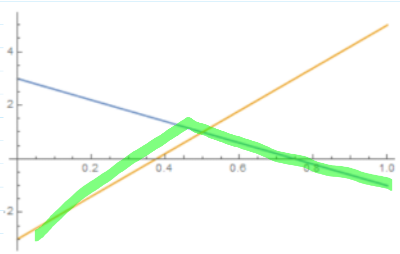
$E(x,2) =$

$E(x,3) =$



$E(1,y) =$

$E(2,y) =$



$A = \begin{pmatrix} 0 & 1 & 3 \\ 3 & 5 & -3 \end{pmatrix}$

Linear Programming

	F	C	S
F	0.30	0.25	0.20
C	0.26	0.33	0.28
S	0.28	0.30	0.33

find $\underset{\sim}{X} = (x_1\ x_2\ x_3)$

Maximize v subject to

$E(X,1) = 0.30x_1 + 0.26x_2 + 0.28x_3 \leq v$

$E(X,2) = 0.25x_1 + 0.33x_2 + 0.30x_3 \leq v$

$E(X,3) = 0.20x_1 + 0.28x_2 + 0.33x_3 \leq v$

$x_1 + x_2 + x_3 = 1$

$x_1, x_2, x_3 \geq 0$

II

minimize v subject to

$E(1,Y) =$

$E(2,Y) =$

$E(3,Y) =$