Equivalence of Games

G is equivalent to G'

For impartial, normal games G, G', say $G \approx G'$ if and only if

for any game H, $G + H$ and $G' + H$ are both N positions or both P positions

Is $*2 \approx *1$? **NO**

Nimbers

$*2 + *2$ P (losing)

$*1 + *2$ N (winning)

winning move to $*1 + *1$ (P position)

Is $*5 \approx *3$? **NO**

$*5 + *3$ N (move $*3 + *3$)

$*3 + *3$ P

Conjecture: $\forall m, n \in \mathbb{N}$, $m \neq n \rightarrow *m \not\approx *n$ (assume $m > n$; winning move is to $\underbrace{*n + *n}_{P}$)

Is $\underbrace{*2 + *1}_{G} \approx \underbrace{*3}_{G'}$

$*2 + *1 + *0$ N $*3 + *0$ N

$*2 + *1 + *1$ N $*3 + *1$ N

$*2 + *1 + *2$ N $*3 + *2$ N

$*2 + *1 + *3$ P $*3 + *3$ P

$*3$ $*3$

$\ast 2 + \ast 1 + \cancel{\ast 1}$ $\qquad$ $\ast 3 + \cancel{\ast 4}$

N $\qquad$ N

Conjecture: $\ast n + \ast m \approx \ast(n \oplus m)$

Properties of Equivalence

For all finite, impartial, normal games G, H, K

$G \approx H \longrightarrow$ G, H have same outcome class

$\underline{G + *0}$ $\underline{H + *0}$ same outcome class
$\quad \shortparallel \qquad \shortparallel$
$\quad \underline{G} \qquad \underline{H}$

equivalence relation

reflexive $G \approx G$

symmetric $G \approx H \longrightarrow H \approx G$

transitive $\underline{G} \approx H$ and $H \approx \underline{K} \longrightarrow G \approx K$

$G + H \approx H + G$ commutative

$(G+H)+K \approx G+(H+K)$ associative

L1: Any position $G+H$ is an N position if G,H are in different outcome classes and is a P position if G,H are both P positions.

$N+P=N$
$P+N=N$
$P+P=P$
$N+N=??$

Proof: (Ind. on length of $G+H$)

Base case $(n=0)$ Then $G+H=\{\}$ and $G=\{\}$ and $H=\{\}$

Ind. step: Suppose $G+H$ has length $k>0$ and suppose all shorter sums satisfy

Three cases: 1) G is N, H is P [want $G+H$ to be N]

then G has option G' s.t. G' is a P position

$\underbrace{G'}_{P}+\underbrace{H}_{P}$ is shorter than $G+H$ so ind. hyp. applies and $G'+H$ is a P pos.

$\therefore$ So $G+H$ has option $G'+H$ which is a P so $G+H$ is N

2) G is P, H is N similar to case 1

3) G,H both P

Then all moves on G are to N (def N)
all moves on H are to N
so all moves on $G+H$ are to $\underbrace{N+P}_{N}$ or $\underbrace{P+N}_{N}$ (def $+$; ind. hyp)
so $G+H$ is a P position (def P)

L2: For every P position A and every position G, $G+A \approx G$

Proof: Suppose A is a P position and G is any position

Let H be any position.

Two cases: 1) $G+H$ is P position. Then $G+A+H \approx \underbrace{G+H}_{P}+\underbrace{A}_{P}$
so $G+A+H$ is P (L1)

2) $G+H$ is N Then $G+A+H \approx \underbrace{G+H}_{N}+\underbrace{A}_{P}$ is N pos (L1)

so $G+H$, $(G+A)+H$ have same outcome class

$G+H$, $(G+A)+H$ have same outcome class for all H

$\therefore$ $G \approx G+A$ (def $\approx$)

L3: $G \approx G'$ if and only if $G+G'$ is a P position

Proof: $\rightarrow$: Suppose $G \approx G'$. Then $G+G$, $G'+G$ have same outcome class
$G'+G = G+G'$

$G+G$ is P, so $G+G'$ is too

$\leftarrow$: Suppose $G+G'$ is a P position

Then $G+(G+G') \approx G$ (L2)

and $G'+(G+G) \approx G'$ (L2)

So $G \approx G+(G+G') \approx (G+G)+G' \approx G'+(G+G) \approx G'$

So $G \approx G + (G + G') \approx (G+G) + G' \approx G' + (G+G) \approx G'$

associative, commutative

and $G \approx G'$ (transitivity)

L4: If $G = \{G_1, \dots, G_n\}$ and $G_1 \approx G_1', \dots, G_n \approx G_n'$ then

$$G \approx \underbrace{\{G_1', \dots, G_n'\}}_{G'}$$

Proof: [use L3: show $G + \{*n_1, \dots, *n_k\}$ is P pos]

Consider options of $G + \underbrace{\{*n_1, \dots, *n_k\}}_{=S}$

move on G or move on S

1) $G_i + \{*n_1, \dots, *n_k\}$, (move on G to one of $G_1, \dots, G_k$)

N pos b/c has option $G_i + *n_i$ where $G_i \approx *n_i$, which is a P pos (L3)

S is set of what options of G are equiv to

G_i is an option of G and is $\approx *n_i$

2) $G + *n_i$, (move on $\{*n_1, \dots, *n_k\}$)

N pos b/c has option $G_i + *n_i$, which is a P pos,

$*n_i \in S$ b/c $*n_i \approx G_i$ for some option G_i of G; use L3

All options are N pos, so $G + \{*n_1, \dots, *n_k\}$ is a P pos

so $G \approx \{*n_1, \dots, *n_k\}$ (L3)

1-row nim w/ some # of stones

Every finite, impartial normal game is equivalent to some nimber.

Proof : (ind. on length of game)

Base case $(n=0)$: only game of length 0 is $\{\} = *0 \approx *0$

Induction step: Let G be a game of length $k>0$ and suppose all games G' of length $<k$ are equivalent to some nimber.

Write $G = \{G_1, \dots, G_\ell\}$ where $len(G_i) < len(G)$ for all i

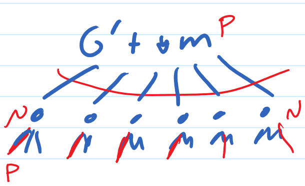
So by induction hypothesis,
$G_1 \approx *n_1,\ G_2 \approx *n_2, \dots, G_\ell \approx *n_\ell$

and $G \approx \underbrace{\{*n_1, *n_2, \dots, *n_\ell\}}_{G'}$

minimum excludant (smallest non-neg not in set)

Claim: $G' + *m$ is P-pos where $m = mex(\{n_1, \dots, n_\ell\})$
so $G' \approx *m$

$G' \approx *m$



Consider all options of $G' + *m$

Three cases: i) $G' + *j$, (move on $*m$) $j < m$
N

$*j + *j$ is an option of $G' + *j$ (mex)
P

Move on G' to $*i$

ii) $*i + *m$, $i < m$
N

winning move is to $*i + *i$
P

iii) $*i + *m$, $i > m$
N

winning move to $*m + *m$
P

~~iv) $*i + *m$, $i = m$~~ (can't happen – mex)

All options of $G' + *m$ are N-positions

So $G' + *m$ in a P position (def)

$\therefore\ G' \approx *m$ (L3)

$G \approx G'$

$G \approx *m$ (transitivity)

Theorem : $*n + *m \approx *(n \oplus m)$

Proof: (induction on length of game, $n+m$)

Base case $(n+m=0)$: Then $n=0,\ m=0,\ n\oplus m=0$
$*n + *m = *0 + *0 = \{\} = *0$

Induction Step: Suppose $n+m>0$ and all n', m' s.t. $n'+m' \le n+m$ have $*n' + *m' \approx *(n'\oplus m')$

$$*n + *m = \{*0 + *m, \dots, *(n-1) + *m,\ *n + *0, \dots, *n + *(m-1)\}$$

$$\approx \{*(0\oplus m), \dots, *((n-1)\oplus m),\ *(n\oplus 0), \dots, *(n\oplus(m-1))\}$$ (ind. hyp, L4)

$$\approx *\mathrm{mex}(\{0\oplus m, \dots, (n-1)\oplus m,\ n\oplus 0, \dots, n\oplus(m-1)\})$$ (Sprague-Grundy)

we can list options of $*n + *m$ and apply ind. hyp.

Claim: $\mathrm{mex}(\{\underbrace{0\oplus m, \dots, (n-1)\oplus m}, n\oplus 0, \dots, n\oplus(m-1)\}) = n\oplus m$

1) $n\oplus m$ is excluded:

in 1st half → suppose $n\oplus m = i\oplus m$, $i<n$
then $n\oplus m\oplus m = i\oplus m\oplus m$
$n = i \Rightarrow\Leftarrow$

in 2nd half → suppose $n\oplus m = n\oplus i$, $i<m$
then $n\oplus n\oplus m = n\oplus n\oplus i$
$m = i \Rightarrow\Leftarrow$

2) All x s.t. $0 \le x < n\oplus m$ are included:

Find most significant bit where $x, n\oplus m$ differ

1) That bit is 1 in $n\oplus m$ and 0 in x (x is smaller)

2) To be 1 in $n\oplus m$, corresponding bits in n, m are $0,1$ or $1,0$

Assume, WLOG, bits are 1 in n, 0 in m

3) So $m\oplus x < n$
and $*(m\oplus x) + *m$ is an option of $*n + *m$

But $*(m\oplus x) + *m \approx *(m\oplus x\oplus m)$ (ind. hyp.)
$= *x$
so $*x$ is included in

2 $\begin{array}{l|l|llll}
n & d_1 & d_2 & \dots & 1 & \dots \\
m & \boxed{e_1} & e_2 & \dots & 0 & \dots \\
\end{array}$

1) $\begin{array}{l|l|llll}
n\oplus m & a_1 & a_2 & \dots & 1 & \dots\ b_k \\
x & \boxed{a_1} & a_2 & \dots & 0 & \dots\ c_k \\
\end{array}$ ← 3)

$\begin{array}{lllll}
m\oplus x & d_1 & d_2 & \dots & 0 \ \dots \\
\end{array}$

3) let's compare n, $m\oplus x$

b) $e_1 \oplus a_1$

a) $= e_1 \oplus d_1 \oplus e_1$
$= d_1$

works everywhere
x, $n\oplus m$ are same,
incl to left of 1st diff

Kayles

0, 1, 2, 3, 1, 4, 3, 2, 1, 4, 2, 6, 4, 1, 2, 7, 1, 4, 3, 2, 1, 4, 6, 7, 4, 1, 2, 8, 5, 4, 7, 2, 1, 8, 6, 7

$\sqcup = \{\ \} \qquad *0$

$x = \{\sqcup\} = \{*0\} \qquad *1$

$x \quad x = \{\underset{*1}{x}\} \qquad *0$

$xx = \{\underset{*1}{x}, \underset{*0}{\sqcup}\} \qquad *2$

01
10
11

$xxx = \{\underset{*2}{xx}, \underset{*0}{x \quad x}, \underset{*1}{x}\} \qquad *3$

$xxxx = \{\underset{*3}{xxx}, \underset{*1 \oplus *2 = *3}{x \quad xx}, \underset{*2}{xx}, \underset{*1 \oplus *1 = *0}{x \quad x}\} \qquad *1$

$xxxxx = \{\underset{*1}{xxxx}, \underset{*3 \oplus *1 = *2}{xxx \quad x}, \underset{*0}{xx \quad xx}, \underset{*3}{xxx}, \underset{*1 \oplus *2 = *3}{x \quad xx}\} \qquad *4$

$xxxxxx = \{\underset{*4}{xxxxx}, \underset{*1}{xxxx}, \underset{*1 \oplus *1 = *0}{x \quad xxxx}, \underset{*1 \oplus *3 = *2}{x \quad xxx}, \underset{*2 \oplus *3 = *1}{xx \quad xxx}, \underset{*2 \oplus *2 = *0}{xx \quad xx}\} \qquad *3$

$\mathrm{mex}(\{4, 1, 0, 2, 1, 0\}) = 3$

xxxxxxxxx xxxxxx x xxxxxxx

???