Mixed Strategies - probability distribution over choices

	R	P	S
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

m rows
n cols

$X = (x_1, \ldots, x_m)$ P I

x_i = prob I plays i

pure strategy = mixed with $x_i = 1$ for one i

$x_1 + \cdots + x_m = 1$

$0 \le x_i \le 1$

$Y = (y_1, \ldots, y_n)$ P II

$y_1 + \cdots + y_n = 1$

$0 \le y_j \le 1$

$X^* = \left(\frac{1}{2}\ \frac{1}{2}\ 0\right) = Y^*$

$$E(X,Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} P(\text{I plays } i \wedge \text{II plays } j) \cdot a_{ij}$$

expected payoff for I given X, Y

$$\sum_{i=1}^{m} \sum_{j=1}^{n} P(\text{I plays } i) \cdot P(\text{II plays } j) \cdot a_{ij}$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \cdot x_i \cdot y_j$$

$$= \underset{1\times m}{X}\ \underset{m\times n}{A}\ \underset{n\times 1}{Y^T}$$

$1\cdot 2 + 0 \cdot 1 = 2$

$$\underset{m\times n}{\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}} \cdot \underset{n\times q}{\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}} = \underset{m\times q}{\begin{pmatrix} 2 & 1 \\ 7 & 2 \end{pmatrix}}$$

$$\underset{1\times 3}{\underset{X}{\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}}} \underset{3\times 3}{\underset{A}{\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}}} \underset{Y^T}{\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}} = \begin{pmatrix} \underset{E(X,1)}{0} & \underset{E(X,2)}{0} & \overset{E(X,3)}{0} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = 0$$

pure strategy with $y_1 = 1$ (for X, pure w/ $x_i = 1$)

(saddle point)

A pair of mixed strategies X^*, Y^* is an equilibrium if

$$E(X,Y^*) \le E(X^*,Y^*) \le E(X^*,Y) \quad \text{for all } X, Y$$

I changes II changes

Every game has a saddle point (equilibrium) in mixed strategies

Equilibrium Theorem: If X^*, Y^* is a saddle point (equilibrium) in mixed strategies for A with $x_i > 0$ $y_j > 0$ then

$$E(X^*, j) = E(i, Y^*) = \text{value}(A)$$

$$\text{value}(A) \rightarrow E(X^*, Y^*)$$

$v(RPS) = E(X^*, Y^*) = E(X^*, j) = E(i, Y^*)$ for all i, j b/c $x_i > 0$, $y_j > 0$

Value of a Game

Suppose there are two saddle points (equilibrium) in mixed strategies (X_1^*, Y_1^*) (X_2^*, Y_2^*)

$$
\begin{aligned}
E(X_1^*, Y_2^*) &\le E(X_2^*, Y_2^*) && \text{I changing, so no better for I} \\
&\le E(X_2^*, Y_1^*) && \text{II changing, so no better for II} \\
&\le E(X_1^*, Y_1^*) \\
&\le E(X_1^*, Y_2^*)
\end{aligned}
$$

=

squeeze all $\le$ are =

Best response: Best response to X is a Y to minimize $E(X, Y)$

Y is an X to maximize $E(X, Y)$

Equilibrium strategies X^*, Y^* are best responses to each other

Finding Saddle Points in Mixed Strategies

Thm: X, Y is an equilibrium in mixed strategies and $\text{value}(A) = E(X, Y)$

if and only if

$$E(i, Y) \le E(X, Y) \le E(X, j) \quad \text{for all } i, j$$

Proof: $\Rightarrow$ from def of equilibrium

$\Leftarrow$ Suppose

Let X', Y' be any mixed strategies w/ $X' = (x_1' \cdots x_m')$ $Y' = (y_1' \cdots y_n')$

Then $x_1' E(1, Y) \le x_1' \cdot E(X, Y)$

$x_2' \cdot E(2, Y) \le x_2' \cdot E(X, Y)$

$\vdots$

$+ \; x_m' E(m, Y) \le x_m' E(X, Y)$

$E(X', Y) \le \cancel{(x_1' + \cdots + x_m')} E(X, Y)$

$E(X', Y) \le E(X, Y)$

$y_1' E(X, Y) \le y_1' \cdot E(X, j)$

$\vdots$

$+ \; y_n' E(X, Y) \le y_n' E(X, n)$

$E(X, Y) \le E(X, Y')$

def of X, Y being equilibrium

	L	C	S
F	0.30	0.25	0.20
C	0.26	0.33	0.28
S	0.28	0.30	0.33

claim: $X^* = (\frac{2}{7} \; 0 \; \frac{5}{7})$ $Y^* = (\frac{5}{7} \; \frac{2}{7} \; 0)$ is an equilibrium

$E(X^*, 1) = \frac{2}{7} \cdot 0.3 + \frac{5}{7} \cdot 0.28 \; \overset{\frac{200}{700}}{\ge} \; E(X^*, Y^*) \quad \frac{2}{7}$ ✓

$E(X^*, 2) = \frac{2}{7} \cdot 0.25 + \frac{5}{7} \cdot 0.30 \; \overset{?}{\ge} \; \frac{2}{7}$ ✓

$E(X^*, 3) = \frac{205}{700} \; \overset{?}{\ge} \; \frac{200}{700} = \frac{2}{7}$ ✓

$E(1, Y^*) \le \frac{2}{7}$? $\quad \frac{2}{7} \le \frac{2}{7}$ ✓

$E(2, Y^*) \le \frac{2}{7}$? $\quad \frac{196}{700} \le \frac{2}{7}$ ✓

$E(3, Y^*) \le \frac{2}{7}$? $\quad \frac{2}{7} \le \frac{2}{7}$ ✓

	L	R
L	$\frac{1}{2}$	1
R	1	$\frac{2}{3}$

$X^* = (x_1, \; 1 - x_1)$

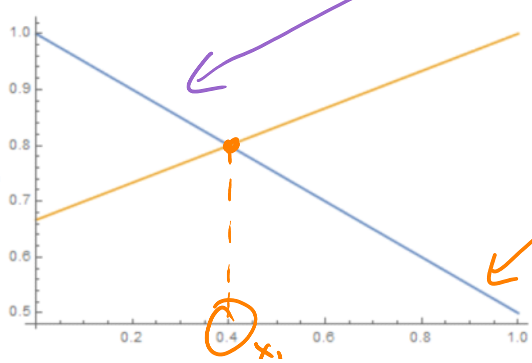
$Y^* = (y_1, \; 1 - y_1)$

$E(X^*, 1) \ge E(X^*, Y^*) = v$ (value of game)

$E(X^*, 2) \ge E(X^*, Y^*)$

$\frac{1}{2} \cdot x_1 + 1 \cdot (1 - x_1) = 1 - \frac{1}{2} x_1 \ge v$

$1 \cdot x_1 + \frac{2}{3} \cdot (1 - x_1) = \frac{2}{3} + \frac{1}{3} x_1 \ge v$



Linear Programming

	F	C	S
F	0.30	0.25	0.20
C	0.26	0.33	0.28
S	0.28	0.30	0.33

$\underline{\text{II}}$

minimize v subject to

$$E(1, Y) =$$

$$E(2, Y) =$$

$$E(3, Y) =$$