For Nim, there is a winning move (N position) if and only if the bitwise exclusive or (Nim-sum) of the number of stones left in each row is non-zero, and the winning moves are the ones that make the bitwise exclusive or 0.

(So a position is an N-position if and only if Nim-sum $\neq 0$)
P-pos iff Nim-sum = 0

100010
001111
101101
000000
xor has become 0 → P position

34 → 100010
~~16~~ 15 ⇒ 010000
45 → 101101

xor 011111
msb

changes to 0
no change
change doesn't matter
010000
$\oplus$ 011111
001111 = 15
less than

non-zero, N position (next player has winning move)

Compute Nim-sum (xor) x
find msb in result
find row w/ msb set
change that row to old value $\oplus$ x ← always legal (reducing # of stones)

for any finite, impartial, normal
convert to equiv form of Nim
compute winning move for Nim
find equiv move in original game

Proof (strong induction on # of stones left (max # moves in game))

Base case ($n=0$): The only game with 0 stones is already over, previous player took last stone and won, so is a P position as required. Nim-sum = 0

Ind step: Suppose $k>0$ and all positions with i stones, $0 \le i < k$ are N-positions iff their Nim-sums are non-zero.

→ Suppose position $m_1, \dots, m_r$ has $m_1 + \dots + m_r = k$ and $m_1 \oplus \dots \oplus m_r = 0$. [want that $m_1, \dots, m_r$ is a P-position]

A move from this position reduces some m_i to m_i' where $m_i' < m_i$

and results in a position with Nim-sum $m_1 \oplus \dots \oplus m_{i-1} \oplus m_i' \oplus m_{i+1} \oplus \dots \oplus m_r$ $\oplus 0$

$= m_1 \oplus \dots \oplus m_{i-1} \oplus m_i' \oplus m_{i+1} \oplus \dots \oplus m_r \oplus m_i \oplus m_i$

$= \underline{m_1 \oplus \dots \oplus m_{i-1} \oplus m_i \oplus m_{i+1} \oplus \dots \oplus m_r} \oplus m_i \oplus m_i'$

$= 0 \oplus m_i \oplus m_i'$

$\neq 0$ since $m_i \neq m_i'$

[want this $\oplus$ to be $\neq 0$ so can apply ind hyp to get result is N pos]

Also, $0 \le m_1 + \dots + m_{i-1} + m_i' + m_{i+1} + \dots + m_r < k = m_1 + \dots + m_r$ (b/c $m_i' < m_i$)
So ind. hyp. applies so $m_1, \dots, m_{i-1}, m_i', m_{i+1}, \dots, m_r$ is N-pos

Move was arbitrary, so all moves lead to N-pos, so original was a P-pos

← Suppose position $m_1, \dots, m_r$ has $m_1 + \dots + m_r = k$ and $m_1 \oplus \dots \oplus m_r = x \neq 0$. [want pos is an N-pos

$\leftarrow$ Suppose position $m_1, \dots, m_r$ has $m_1 + \cdots + m_r = k$ and $m_1 \oplus \cdots \oplus m_r = x \neq 0$. [want pos is an N-pos $\Leftrightarrow$ is some move to a P-pos]

Find m_i s.t. most significant bit of x is 1 is m_i (must be one or msb not 1 in x)
Reduce m_i stones to $m_i' = m_i \oplus x$ (this is a reduction b/c msb changes $1 \mapsto 0$ and bits to left don't change)

Nim-sum of result is $m_1 \oplus \cdots \oplus m_{i-1} \oplus m_i' \oplus m_{i+1} \oplus \cdots \oplus m_r$

$$= m_1 \oplus \cdots \oplus m_{i-1} \oplus m_i \oplus x \oplus m_{i+1} \oplus \cdots \oplus m_r$$

$$= m_1 \oplus \cdots \oplus m_{i-1} \oplus m_i \oplus m_{i+1} \oplus \cdots \oplus m_r \oplus x$$

$$= x \oplus x$$

$$= 0$$

And since $m_1 + \cdots + m_{i-1} + m_i' + m_{i+1} + \cdots + m_r < m_1 + \cdots + m_r$, the induction hypothesis applies
So the resulting position $m_1, \dots, m_{i-1}, m_i', m_{i+1}, \dots, m_r$ is a P-pos b/c its Nim-sum $= 0$

Since there is a move from $m_1, \dots, m_r$ to a P-pos, $m_1, \dots, m_r$ is an N-pos as required]

Using Sprague-Grundy

Sprague-Grundy Theorem: every finite, normal, impartial combinatorial game is equivalent to some form of 1-row Nim.

Corollary: If G is equivalent to *n and H is equivalent to *m then G+H is equivalent to *(n⊕m)

set of equiv Nim positions — min excluded — Numbers

Kayles	position	Reachable		set of equiv Nim positions	mex(Reachable)	Grundy
␣	K0	{ }	≈	{ }	0	*0 (0 Nim stones)
x	K1	{␣} = {K0}	≈	{*0}	1	*1
xx	K2	{x, ␣}	≈	{*1, *0}	2	*2
x x		{x}	≈	{*1}	0	*0
xxx	K3	{xx, x.x, x}	≈	{*2, *0, *1}	3	*3
xxxx	K4	{xxx, xx.x, xx, x.x}				

$= \{K3, K2+K1, K2, K1+K1\}$

$\approx \{*3, *2+*1, *2, *1+*1\}$

$*n + *m \approx *(n \oplus m)$

$\approx \{*3, *3, *2, *0\}$ — 1 — $*1$

xxxxx K5 $\{K4, K1+K3, K2+K2, K3, K2+K1\}$ — $*4$

$\approx \{*1, *1+*3, *2+*2, *3, *2+*1\}$

$\approx \{*1, *2, *0, *3, *3\}$

reduce this to $*0$

x xx xxx xxxxx $\approx K1 + K2 + K3 + K5$

$\approx *1 + *2 + *3 + *4$

$\approx *(1 \oplus 2 \oplus 3 \oplus 4)$

$= *4$ N-position

need to change this to something $\approx *0$

(search through moves)

.xxxx $\approx *1$ NO

~~x.xxx $\approx *1 + *3$~~ NO

xx.xx $\approx *2 + *2$

$\approx *0$ YES

```
  001
  010
  011
  100  -> 0
  ---
  100
  msb
```

```
100
100
---
000
```

for any finite, impartial, normal game written as a sum
- compute Grundy numbers up to largest game in sum
- convert each term to Nim according to those Grundy numbers
- find a winning Nim move
- find corresponding move in original game

K1 + K2 + K3 + K5

[0, 1, 2, 3, 1, 4]

$*1 + *2 + *3 + *4$

$*4 \to *0$

xxxxx → xx.xx

Game position = set of options
game = position   pos reachable in one move

In traditional 1-row Nim   Nimbers

⌴ = $*0$

O = $*1$

O O = $*2$

O O O = $*3$

O O O O = $*4$

Outcome class = who has winning strategy   N – (current) next player
P – previous

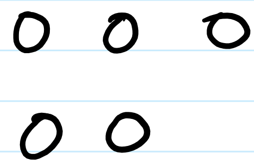
$\{\} = *0$ is a P position (for normal games)

position G is an N position if $\exists$ option G' s.t. G' is P-position

P position if $\forall$ options G', G' is N-position

in a finite, normal, impartial game all positions are P or N

Sums of Games

 $= *3 + *2$

$$G + H = \{ G' + H \mid G' \text{ is an option of } G \} \cup \{ G + H' \mid H' \text{ is an option of } H \}$$

Equivalence of Games

For impartial normal games G, G', say $G \approx G'$ if and only if for all other games H, $G+H$ and $G'+H$ have same outcome class

Is $*2 \approx *1$?

$*2 + \underbrace{*2}_{H}$ P pos (mirroring)

$*1 + \underbrace{*2}_{H}$ N pos (winning move goes to $*1+*1$)

Is $*5 \approx *3$?

$*5 + \underbrace{*3}_{H}$ N pos (move to $*3+*3$)

$*3 + \underbrace{*3}_{H}$ P pos

Conjecture: $\forall m,n \in \mathbb{N},\ m \neq n \longrightarrow *m \not\approx *n$ (prove by showing $*m + \underline{*n}$ is N-pos, $*m + *m$ is P-pos)

Is $*2 + *1 \approx *3$

$*2+*1+\underline{*0}$ N	$*3+\underline{*0}$ N
$*2+*1+\underline{*1}$ N	$*3+\underline{*1}$ N
$*2+*1+\underline{*2}$ N	$*3+\underline{*2}$ N
$*2+*1+\underline{*3}$ P	$*3+\underline{*3}$ P

YES? $*2+*1 \approx *3$

$*2+*1+\_\_\_$ $*3+\_\_\_$

Conjecture: $*n + *m \approx *(n \oplus m)$ (consequence of Sprague-Grundy)