Normal Kayles:

× × × × × × × + × × + × × × × + × × × × × × × × × × + ×

2   2   4   10   1

. × × × × × × ≈ $*3$

× . × × × × × ≈ $*1 + *4 ≈ *5$

× × . × × × × ≈ $*2 + *1 ≈ *3$

× × × . × × ×
≈ $*3 + *3$
≈ $*0$

$*2 + *2 + *1 + *2 + *1$

$2 \oplus 2 \oplus 1 \oplus 2 \oplus 1 = 2 \neq 0$ so N-pos

→ ~~10~~ 00

10
10
00

10
01
10
01
10

msb

$0 \oplus 2 \oplus 1 \oplus 2 \oplus 1 = 0$ so P-pos

Properties of Equivalence

For all finite, impartial, normal games G, H, K

$G \approx H \longrightarrow$ G, H have same outcome class

$G+\{\}$ $\quad H+\{\}$

equiv relation

reflexive $\quad G \approx G$

symmetric $\quad G \approx H \longrightarrow H \approx G$

transitive $\quad G \approx H$ and $H \approx K \longrightarrow G \approx K$

from "has same outcome class" being an equiv relation

commutative $\quad G+H \approx H+G$

associative $\quad (G+H)+K \approx G+(H+K) \approx G+H+K$

Lemmas

$\forall n \in \mathbb{Z}$, if $n \geq a$, $P(n)$

L1: Any position $G+H$ is an N position if G, H are in different outcome classes and is a P position if G, H are both P positions. ☆

Proof: (Induction on length of game): for all $l \geq 0$, for $G+H$ of length l, ...

Base case: ($l = 0$) then $G+H = \{\} = *0$ (P pos) and $G = \{\} = H = *0$ (P pos)

Ind step: Suppose $G+H$ has length $k > 0$ and suppose sums of length $< k$ satisfy ☆

3 cases

1) G is N-pos, H is P-pos

want $G+H$ is N-pos (has option that is P-pos)

Since G is N-pos, has opt G' that is P (def N)

$G+H$ has option $G'+H$ that is P-pos + P-pos where $\text{len}(G'+H) < k$ so ind. hyp. applies so $G'+H$ is a P pos

$\therefore G+H$ is N pos (def N: has option that is P)

2) G is P-pos, H is N-pos

symmetric

3) G, H both P-pos

want $G+H$ is P-pos

similar - consider all options of $G+H$

$G'+H$ — $N+P$

or $G+H'$ — $P+N$

apply ind hyp to get option is N

$\therefore G+H$ P pos (all opts are N)

L2: For every P position A and every position G, $G+A \approx G$

Proof: Suppose A is a P position and G is any position [want $G+A+H$, $G+H$ have same outcome class for all H]

Two cases: 1) $G+H$ is N-pos [want $G+A+H$ is N pos]

Then $G+A+H \approx \underbrace{(G+H)+A}_{N\ +\ P}$

where $(G+H)+A$ is an N-pos

so $G+A+H$ is too since it is $\approx (G+H)+A$

2) $G+H$ is P pos [want $G+A+H$ is P-pos]

similar

L3: $G \approx G'$ if and only if $G + G'$ is a P position

Proof: $\rightarrow$: Suppose $G \approx G'$. Then $G + \underset{H}{\underline{G}}$ and $G' + \underset{H'}{\underline{G}}$ have same outcome class.

where $G+G$ is a P pos

so $G + G'$ is too since $G \approx G'$

$\leftarrow$: Suppose $G + G'$ is a P position

Then
$$\underline{G} + \underset{A}{\underline{G + G'}} \approx G$$
$$\text{SS}$$
$$G' + \underset{A}{\underline{G + G}} \approx G'$$

lemma 2

so $G \approx G'$

Every finite, impartial normal game is equivalent to some nimber.

Proof: (induction on length of game)

Base case ($n=0$): only len 0 game $\{\} = *0 \approx *0$

Induction step: Let G be a game of length $k>0$ and suppose all games G' of length $<k$ are equivalent to some nimber.

Write $G = \{G_1, \dots, G_\ell\}$ ($\text{len}(G_i) < k$ for all i)

So by induction hypothesis, can find $n_1, \dots, n_\ell$ s.t. $G_1 \approx *n_1, \dots, G_\ell \approx *n_\ell$

$G + G'$ is P pos (so by L3 $G \approx G'$) ⟵ So $G \approx \underbrace{\{*n_1, *n_2, \dots, *n_\ell\}}_{G'}$

Claim: $G' + *m$ is P-pos where $m = \text{mex}(\{n_1, \dots, n_\ell\})$ so $G' \approx *m$ (by L3) and $G \approx *m$ (trans)

Two cases: 1) $G' + *m = \{\} = *0$
no options
then $G' = *m = \{\}$ and $\{\}$ is a P-pos

2) Consider all options of $G' + *m$
Three cases: i) $G' + *i$, $i < m$
there are options
$\underbrace{*i + *i}_{\text{P-pos}}$ is an option of $G' + *i$ (m is mex, $i < m$)
so $G' + *i$ is an N-pos

ii) $*i + *m$, $i < m$
then $*i + *i$ is an option of $*i + *m$
since $*i$ is an option of $*m$
$*i + *i$ is a P-pos
so $*i + *m$ is an N-pos

iii) $*i + *m$, $i > m$
move to $*m + *m$
so $*i + *m$ N-pos

iv) ~~$*m + *m$~~ (m is mex)

All options of $G' + *m$ are N-positions

$G' + *m$ is P pos

so by L3 $G' \approx *m$

also, $G \approx G'$, so $G \approx *m$ (by transitivity)

Nim-sum

Theorem : $*n + *m \approx *(n \oplus m)$

Proof: (Induction on $n+m$) $\forall k \geq 0, \forall n,m$ s.t. $n+m=k$, ...

Base case ($n+m=0$). Then $n=0, m=0, n\oplus m = 0$
$*n + *m = *0 + *0 = \{\} = *0$

Induction Step: Suppose $n+m>0$ and all n', m' s.t. $n'+m' \leq n+m$ have $*n' + *m' \approx *(n' \oplus m')$

$$*n + *m = \{*0+*m, *1+*m, \dots, *(n-1)+*m, *n+*0, \dots, *n+*(m-1)\}$$

$$\approx \{*(0\oplus m), \dots, *(n-1 \oplus m), *(n \oplus 0), \dots, *(n\oplus m-1)\} = \Theta$$

$$*n + *m \approx \text{mex}(\{0\oplus m, \dots, n-1\oplus m, n\oplus 0, \dots, n\oplus m-1\})$$

Claim: $\text{mex}(\{0\oplus m, \dots, n-1\oplus m, n\oplus 0, \dots, n\oplus m-1\}) = n\oplus m$

[must show 1) $n\oplus m$ is excluded
2) nothing smaller is excluded]

$n\oplus m$ is excluded : Let $x \in \Theta$ then either
i) $x = n'\oplus m$ for some $0 \leq n' < n$
Suppose $x = n'\oplus m = n\oplus m$
$n' \oplus m \oplus m = n \oplus m \oplus m$
$n' \oplus 0 = n \oplus 0$
$n' = n$ $\rightarrow\leftarrow$
So $x \neq n\oplus m$

ii) $x = n\oplus m'$ for $0 \leq m' < m$
similar: $x \neq n\oplus m$

So $x \neq n\oplus m$
$n\oplus m \notin \Theta$

$*5 + *7 = \{*0+*7, *1+*7, *2+*7, *3+*7, *4+*7, *5+*0, *5+*1, *5+*2, *5+*3, *5+*4, *5+*5, *5+*6\}$
$\approx \{*7, *6, *5, *4, *3, *5, *4, *7, *4, *1, *0, *3\}$

$5 \oplus \_ = 2$ only thing that works is the other
$\_ \oplus 7 = 2$

which option is $\approx *1$?

$5 \oplus (5\oplus 1)$
$7 \oplus (7 \oplus 1)$

All x s.t. $0 \leq x < n\oplus m$ are included:

Find most significant bit where $x, n\oplus m$ differ ($x \neq n\oplus m$)

That bit is 1 in $n\oplus m$ and 0 in x ($x < n\oplus m$)

To be 1 in $n\oplus m$, corresponding bits in n, m are 1,0 or 0,1

Assume, WLOG, bits are 1 in n, 0 in m

So $\underline{m\oplus x < n}$
and $*m\oplus x + *m$ is an option of $*n + *m$

But $*m\oplus x + *m \approx *(m\oplus x \oplus m)$ (ind hyp)

Let $x < n\oplus m$ then either
$n \oplus x < m$ or
$m \oplus x < n$

n	$a_1 a_2 \dots 1 \dots a_n$
m	$b_1 b_2 \dots 0 \dots b_n$
$n\oplus m$	$(a_1\oplus b_1)(a_2\oplus b_2) \dots 1 \dots (a_n \oplus b_n)$

$m \quad \underline{b_1 b_2 \ldots 0 \ldots b_n}$

$n \oplus m \quad (a_1 \oplus b_1)(a_2 \oplus b_2)\ 1 \ldots (a_n \oplus b_n)$

$x \quad (a_1 \oplus b_1)(a_2 \oplus b_2)\ 0 \ldots\ldots$

$m \oplus x \quad a_1 a_2 \ldots 0 \ \ldots$

and $*m \oplus x + *m$ is an option of ...

But $*m \oplus x + *m \approx *(m \oplus x \oplus m)$ (ind hyp)

$= *x$

so $x \in \Theta$ (not excluded)