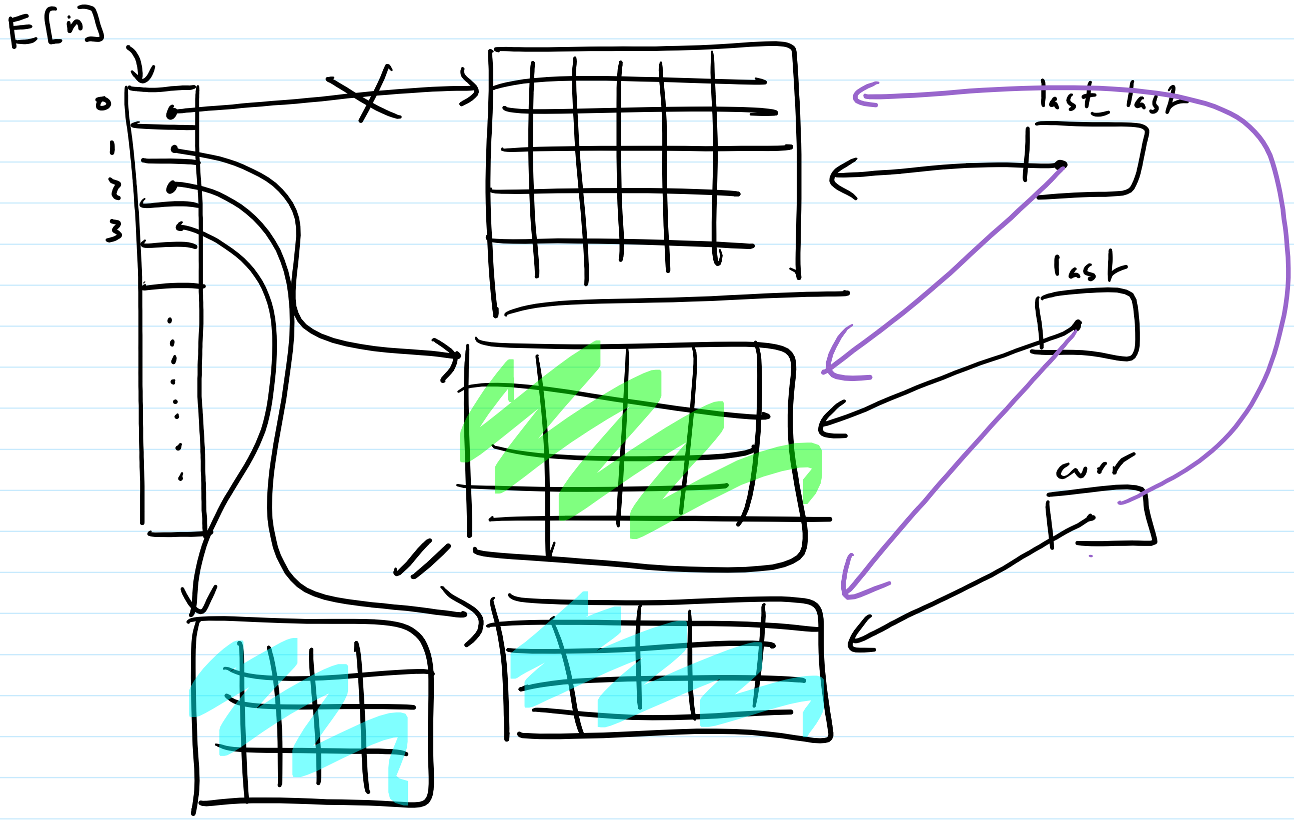

E[n]
0
1
2
3
last_last
last
curr

Simultaneous Play Games

		II		
		R	P	S
I	Rock	0,0	-1,1	1,-1
	Paper	1,-1	0,0	-1,1
	Scissors	-1,1	1,-1	0,0

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

a_{ij} = payout for I when I chooses i, II chooses j
b_{ij} = payout for II

Penalty Kick

		goal keeper L	R
Shooter	L	$\frac{1}{10}, -\frac{1}{10}$	$\frac{1}{2}, -\frac{1}{2}$
	R	$\frac{1}{2}, -\frac{1}{2}$	$\frac{1}{8}, -\frac{1}{8}$

zero-sum : $a_{ij} + b_{ij} = 0$ (so only need A)

constant-sum : $a_{ij} + b_{ij} = C$

const-sum game w/ constant C

1) play zero-sum game $A - C$
2) award player I C

so strategy for constant-sum = strategy for zero-sum

def

off		blitz W	balanced X	pass Y	run Z
run	A	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{2}{4}$
short pass	B	$-\frac{1}{10}$	$\frac{1}{3}$	$\frac{1}{4}$	$-\frac{1}{2}$
long pass	C	$\frac{1}{2}$	0	$-\frac{1}{10}$	$\boxed{\frac{1}{2}}$

expected change in win probability

Stag Hunt

	S	H
Stag	2,2	0,1
Hare	1,0	1,1

non-constant sum game

	R	P	S	row min
Rock	0	-1	1	-1
Paper	1	0	-1	-1
Scissors	-1	1	0	-1
	1	1	1	

$v^+ = 1$ $\quad v^- = -1$

v^- = amount I guaranteed to win
$\max_i \min_j a_{ij}$

no saddle points

	L	R
L	$\frac{1}{2}$	1
R	1	$\frac{2}{3}$

	W	X	Y	Z	row min
A	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{2}{4}$	$\frac{1}{2}$
B	-1	$\frac{1}{3}$	$\frac{1}{4}$	$-\frac{1}{2}$	-1
C	$\frac{1}{2}$	0	-1	$\frac{1}{2}$	-1
col max	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$v^- = \frac{1}{2}$

v^+ = ceiling on what I loses $\quad v^+ = \frac{1}{2}$
$\min_j \max_i a_{ij}$

saddle point

For any constant-sum game, $v^- \le v^+$

$\forall i,j \quad \min_{j'} a_{ij'} \le a_{ij}$ (min in row i always $\le$ any entry in that row)

$\forall j \quad \max_i \min_{j'} a_{ij'} \le \max_i a_{ij}$ (max of smaller things $\le$ max of larger things)

$\min_j \underline{\max_i \min_{j'} a_{ij'}} \le \min_j \max_i a_{ij}$ (min of smaller things $\le$ min of larger things)

$\max_i \min_j a_{ij} \le \min_j \max_i a_{ij}$ (constant w/respect to j; rename inner var $j' \to j$)

$v^- \le v^+$ (def v^-, v^+)

Saddle point: where neither player has incentive to unilaterally change strategy (equilibrium)

i^*, j^* is saddle point means $a_{ij^*} \le a_{i^*j^*} \le a_{i^*j}$

each player chooses a single row/column

A constant-sum game A has a saddle point in <u>pure strategies</u> if and only if $v^- = v^+$

$\Rightarrow$: Suppose A has a saddle point in pure strategies.

Then $\exists\, i^*, j^*$ s.t. $\forall i,j \quad a_{ij^*} \le a_{i^*j^*} \le a_{i^*j}$

so $\max_i a_{ij^*} \le a_{i^*j^*} \le \min_j a_{i^*j}$ (max of small things small, min of big things big)

and $\min_j \max_i a_{ij} \le \max_i a_{ij^*}$ $\min_j a_{i^*j} \le \max_i \min_j a_{ij}$
(min includes $j = j^*$; max includes $i = i^*$)
min $\le$ each term; max $\ge$ each term)

so $v^+ \le \max_i a_{ij^*} \le a_{i^*j^*} \le \min_j a_{i^*j} \le v^-$ (def v^+, v^-; transitivity)

but $v^- \le v^+$ so all $\le$ are $=$ (squeeze)

$\Leftarrow$ Suppose $v^- = v^+$. Let i^* be the i s.t. $v^- = \max_i \min_j a_{ij}$
j^* be the j s.t. $v^+ = \min_j \max_i a_{ij}$

$a_{i^*j} \ge \min_j a_{i^*j} = v^+ = v^- = \max_i a_{ij^*} \ge a_{ij^*}$
(min is $\le$ each term; max is $\ge$ each term)

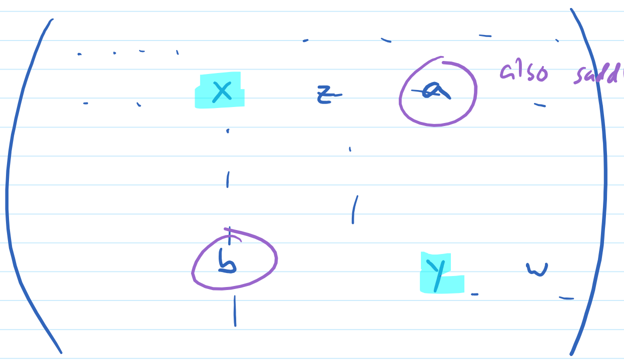
$a_{i^*j^*} \ge v^+ = v^- \ge a_{i^*j^*}$
and $\ge$ are $=$ (squeeze)

$\therefore$ $v^+ = v^- = a_{i^*j^*}$

$a_{ij^*} \le a_{i^*j^*} \le a_{i^*j}$ for all i, j
(substitution)

Suppose there are 2 saddle points in pure strategies (i_1, j_1) and (i_2, j_2)
with values $a_{i_1j_1} = v_1$ $a_{i_2j_2} = v_2$

then $v_1 = v_2 = v^- = v^+$
call this v, value of game



also saddle points

$x \le a \le y \le b \le x$

since $x = 1$
all $\le$ are $=$

$x = a = y = b = x$
so
$x = y$

Mixed Strategies - probability distribution

	R	P	S
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

$(x_1, \ldots, x_n)$ I plays row i w/ prob x_i

$(y_1, \ldots, y_m)$ II play col j w/ prob y_j

guess $X^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = Y^*$