Mixed Strategies – probability distribution

	R	P	S
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

$(x_1, \dots, x_n)$ I plays row i w/ prob x_i

$(y_1, \dots, y_m)$ II play col j w/ prob y_j

$0 \le x_i, y_i \le 1$

$\sum x_i = 1 \quad \sum y_i = 1$

$$X^* = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right) = Y^*$$

$$E(X, Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} \cdot P(\text{I plays row } i, \text{ II plays row } j)$$

$$= \sum \sum a_{ij} \cdot P(\text{I plays } i) \cdot P(\text{II plays } j)$$

$$= \sum \sum a_{ij} \cdot x_i \cdot y_j$$

$$= X A Y^T$$

$$E(X^*, Y^*) = \underbrace{\left(\tfrac{1}{3} \;\; \tfrac{1}{3} \;\; \tfrac{1}{3}\right) \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}}_{(0 \;\; 0 \;\; 0)} \begin{pmatrix} \tfrac{1}{3} \\ \tfrac{1}{3} \\ \tfrac{1}{3} \end{pmatrix} = 0$$

	R	P	S
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

$$X = \left(0 \;\; \tfrac{1}{2} \;\; \tfrac{1}{2}\right)$$

$$Y = (0 \;\; 0 \;\; 1)$$

$$\underbrace{\left(0 \;\; \tfrac{1}{2} \;\; \tfrac{1}{2}\right) \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}}_{} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} =$$

$$\left(0 \;\; \tfrac{1}{2} \;\; -\tfrac{1}{2}\right) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -\tfrac{1}{2}$$

X^*, Y^* is a saddle point in mixed strategies if and only if

$$E(X, Y^*) \le E(X^*, Y^*) \le E(X^*, Y) \quad \text{for all } X, Y$$

Every game has a $\underset{\text{Nash equilibrium}}{\underline{\text{saddle point}}}$ in mixed strategies

Equilibrium Theorem: If X^*, Y^* is a saddle point in mixed strategies for A with $x_i > 0$ $y_j > 0$ then

$$E(X^*, \underbrace{j}_{\substack{\text{pure strategy —}\\ \text{always pick}\\ \text{column } j}}) = E(\underbrace{i}_{\substack{\text{always}\\ \text{pick}\\ \text{row } i}}, Y^*) = \underline{\text{value}(A)}$$

$$= E(X^*, Y^*)$$

$$X^* = \left(\tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{1}{3}\right) \qquad Y = \left(\tfrac{1}{2}\ 0\ \tfrac{1}{2}\right) \qquad E(X^*, Y) = 0$$

$$E(X^*, Y) = \tfrac{1}{2} E(X^*, 1) + 0 \cdot E(X^*, 2) + \tfrac{1}{2} \cdot E(X^*, 3)$$

$$= \tfrac{1}{2} \cdot \text{value}(A) + 0 \cdot \text{value}(A) + \tfrac{1}{2} \cdot \text{value}(A)$$

$$= \left(\tfrac{1}{2} + 0 + \tfrac{1}{2}\right) \cdot \text{value}(A)$$

$$= \text{value}(A) = 0$$

Best Response: Best response to a mixed strategy X is Y that minimizes XAY^T

$$X = \left(0\ \tfrac{1}{2}\ \tfrac{1}{2}\right)$$

$$Y = (0\ 0\ 1) \quad \text{best response to } X$$

Strategies at equilibrium are best responses to each other

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Finding Saddle Points in Mixed Strategies

Thm: X^*, Y^* is a saddle point in mixed strategies and value$(A) = E(X^*, Y^*)$ if and only if

$$E(i, Y^*) \le E(X^*, Y^*) \le E(X^*, j) \quad \text{for all } i, j$$

$\Rightarrow$ Suppose X^*, Y^* is a saddle point. For any i, j let $X = (0 \cdots \underset{x_i}{1} \cdots 0)$, $Y = (0 \cdots \underset{y_j}{1} \cdots 0)$

By def of saddle point, $E(i, Y^*) = E(X, Y^*) \le E(X^*, Y^*) \le E(X^*, Y) = E(X^*, j)$ as required

$\Leftarrow$ Call saddle point (we know there is one) $X_0, Y_0 = (y_1 \cdots y_m)$, $\hookrightarrow = (x_1 \dots x_n)$

$$\begin{aligned}
y_1 E(X^*, Y^*) &\le y_1 E(X^*, 1) \\
y_2 E(X^*, Y^*) &\le y_2 E(X^*, 2) \\
&\vdots \\
y_m E(X^*, Y^*) &\le y_m E(X^*, m)
\end{aligned}$$

$$\begin{aligned}
x_1 E(1, Y^*) &\le x_1 E(X^*, Y^*) \\
&\vdots \\
x_n E(n, Y^*) &\le x_n E(X^*, Y^*)
\end{aligned}$$

$(y_1 + \cdots + y_m) E(X^*, Y^*) \le y_1 \cdot E(X^*, 1) + \cdots + y_m E(X^*, m)$

$1 \cdot E(X^*, Y^*) \le E(X^*, Y_0) \le E(X_0, Y_0) \le$; $E(X_0, Y^*) \le E(X^*, Y^*)$

didn't use the fact that Y_0 is from saddle point, so works for any Y

def saddle point

similar

all $\le$ are $=$ (squeeze)

so $E(X^*, Y^*) = E(X_0, Y_0) = $ value(A) and X^*, Y^* is a saddle point

pitcher

batter	F	C	S
F	0.30	0.25	0.20
C	0.26	0.33	0.28
S	0.28	0.30	0.33

$X^* = (\frac{2}{7}\ 0\ \frac{5}{7})$ $\quad Y^* = (\frac{5}{7}\ \frac{2}{7}\ 0)$

$E(X^*, Y^*) = \frac{2}{7} = X^* \cdot A \cdot Y^{*T}$

Is $E(X^*, 1) = \frac{2}{7} \cdot .30 + \frac{5}{7} \cdot .28 = \frac{200}{700} \ge \frac{2}{7}$ ✓

$E(X^*, 2) = \frac{2}{7} \cdot .25 + \frac{5}{7} \cdot .30 = \frac{200}{700} \ge \frac{2}{7}$ ✓

$E(X^*, 3) = \frac{2}{7} \cdot .2 + \frac{5}{7} \cdot .33 = \frac{205}{700} \ge \frac{2}{7}$ ✓

Is $E(1, Y^*) = \frac{2}{7} \le \frac{2}{7}$ ✓

$E(2, Y^*) = \frac{196}{700} \le \frac{2}{7}$ ✓

$E(3, Y^*) = \frac{2}{7} \le \frac{2}{7}$ ✓

So X^*, Y^* is a saddle point

Suppose there are two saddle points in mixed strategies (X_1^*, Y_1^*) (X_2^*, Y_2^*)

$$\begin{aligned}
E(X_1^*, Y_2^*) &\le E(X_2^*, Y_2^*) \\
&\le E(X_2^*, Y_1^*) \\
&\le E(X_1^*, Y_1^*) \\
&\le E(X_1^*, Y_2^*)
\end{aligned}$$

Finding Solutions for 2x2 payoff matrix

	L	R
L	$\frac{1}{2}$	1
R	1	$\frac{2}{3}$

$(x_1\ x_2) = (x_1\ \ 1 - x_1)$

For saddle point $X^*, Y^* \to (y_1\ \ 1 - y_1)$

$E(X^*, 1) \ge E(X^*, Y^*) = v$

$x_1 \cdot \frac{1}{2} + (1 - x_1) \cdot 1 = 1 - \frac{1}{2}x_1 \ge v$

I wants to maximize v

$E(X^*, 2) \ge E(X^*, Y^*) = v$

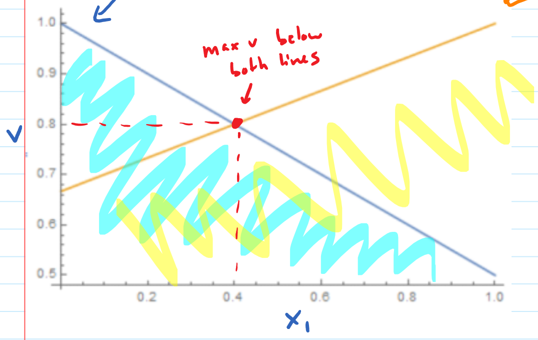
$x_1 \cdot 1 + (1 - x_1) \cdot \frac{2}{3} = \frac{2}{3} + \frac{1}{3}x_1 \ge v$

$E(1, Y^*) \le v \quad \frac{1}{2}y_1 + 1(1 - y_1) = 1 - \frac{1}{2}y_1 \le v$

$E(1, y^*) \le v \qquad \frac{1}{2}y_1 + 1(1-y_1) = 1 - \frac{1}{2}y_1 \le v$

$E(2, y^*) \le v \qquad 1\cdot y_1 + \frac{2}{3}(1-y_1) = \frac{2}{3} + \frac{1}{3}y_1 \le v$

(same graph as for x_1 vs v)



intersection gives $x_1 = \frac{2}{5}$ so $x_2\ \frac{3}{5}$ $\quad v = \frac{4}{5}$

and $\quad y_1 = \frac{2}{5} \quad y_2 = \frac{3}{5}$

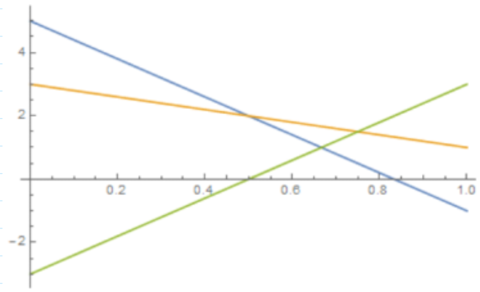
$x^* = (\frac{2}{5}\ \frac{3}{5}) = y^*$ is saddle point

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \end{pmatrix}$$

$E(x,1) =$

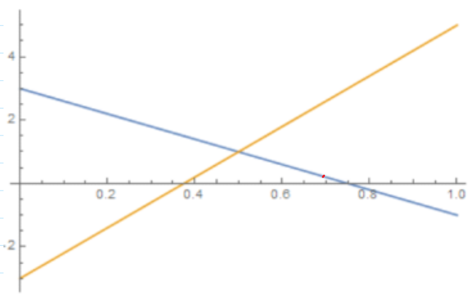
$E(x,2) =$

$E(x,3) =$



$E(1,y) =$

$E(2,y) =$



$$A = \begin{pmatrix} 0 & -1 & 3 \\ 3 & 5 & -3 \end{pmatrix}$$