Finding Saddle Points in Mixed Strategies

Thm: X^*, Y^* is a saddle point in mixed strategies and value$(A) = E(X^*, Y^*)$ if and only if

$$E(i, Y^*) \le E(X^*, Y^*) \le E(X^*, j) \quad \text{for all } i, j$$

$\Rightarrow$ Suppose X^*, Y^* is a saddle point. For any i, j let $X = (0 \cdots \underset{x_i}{1} \cdots 0)$, $Y = (0 \cdots \underset{y_j}{1} \cdots 0)$

By def of saddle point, $\underline{E(i, Y^*)} = E(X, Y^*) \le \underline{E(X^*, Y^*)} \le E(X^*, Y) = \underline{E(X^*, j)}$ as required

$\Leftarrow$ Call saddle point (we know there is one) $X_0, Y_0 = (y_1 \cdots y_m)$, $X_0 = (x_1 \dots x_n)$

$$\begin{aligned}
y_1 E(X^*, Y^*) &\le y_1 E(X^*, 1) \\
y_2 E(X^*, Y^*) &\le y_2 E(X^*, 2) \\
&\vdots \\
y_m E(X^*, Y^*) &\le y_m E(X^*, m)
\end{aligned}$$

$$(y_1 + \cdots + y_m) E(X^*, Y^*) \le y_1 \cdot E(X^*, 1) + \cdots + y_m E(X^*, m)$$

$$1 \cdot E(X^*, Y^*) \le E(X^*, Y_0) \le E(X_0, Y_0) \le$$

didn't use the fact that Y_0 is from saddle point, so works for any Y

def saddle point

$$\begin{aligned}
x_1 E(1, Y^*) &\le x_1 E(X^*, Y^*) \\
&\vdots \\
x_n E(n, Y^*) &\le x_n E(X^*, Y^*)
\end{aligned}$$

$$E(X_0, Y^*) \le E(X^*, Y^*)$$

similar

all $\le$ are $=$ (squeeze)

so $E(X^*, Y^*) = E(X_0, Y_0) = $ value(A) and X^*, Y^* is a saddle point

pitcher

batter	F	C	S
F	0.30	0.25	0.20
C	0.26	0.33	0.28
S	0.28	0.30	0.33

$X^* = (\frac{2}{7} \;\; 0 \;\; \frac{5}{7})$ $\quad Y^* = (\frac{5}{7} \;\; \frac{2}{7} \;\; 0)$

$$E(X^*, Y^*) = \frac{2}{7} = X^* \cdot A \cdot Y^{*T}$$

Is $E(X^*, 1) = \frac{2}{7} \cdot .30 + \frac{5}{7} \cdot .28 = \frac{200}{700} \ge \frac{2}{7}$ ✓

$E(X^*, 2) = \frac{2}{7} \cdot .25 + \frac{5}{7} \cdot .30 = \frac{200}{700} \ge \frac{2}{7}$ ✓

$E(X^*, 3) = \frac{2}{7} \cdot .2 + \frac{5}{7} \cdot .33 = \frac{205}{700} \ge \frac{2}{7}$ ✓

Is $E(1, Y^*) = \frac{2}{7} \le \frac{2}{7}$ ✓

$E(2, Y^*) = \frac{196}{700} \le \frac{2}{7}$ ✓

$E(3, Y^*) = \frac{2}{7} \le \frac{2}{7}$ ✓

So X^*, Y^* is a saddle point

Suppose there are two saddle points in mixed strategies (X_1^*, Y_1^*) (X_2^*, Y_2^*)

$$\begin{aligned}
E(X_1^*, Y_2^*) &\le E(X_2^*, Y_2^*) \\
&\le E(X_2^*, Y_1^*) \\
&\le E(X_1^*, Y_1^*) \\
&\le E(X_1^*, Y_2^*)
\end{aligned}$$

Finding Solutions for 2x2 payoff matrix

	L	R
L	$\frac{1}{2}$	1
R	1	$\frac{2}{3}$

$(x_1 \;\; x_2) = (x_1 \;\; 1 - x_1)$

For saddle point $X^*, Y^* \rightarrow (y_1 \;\; 1 - y_1)$

$E(X^*, 1) \ge E(X^*, Y^*) = v$

$x_1 \cdot \frac{1}{2} + (1 - x_1) \cdot 1 = 1 - \frac{1}{2} x_1 \ge v$

I wants to maximize v

$E(X^*, 2) \ge E(X^*, Y^*) = v$

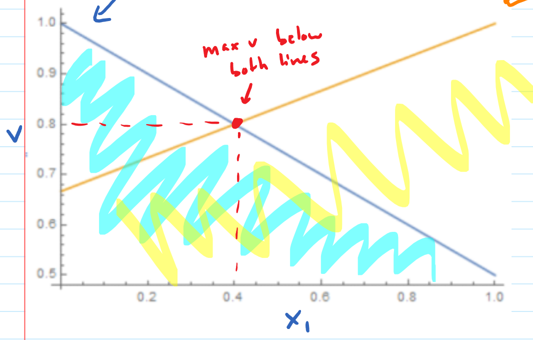
$x_1 \cdot 1 + (1 - x_1) \cdot \frac{2}{3} = \frac{2}{3} + \frac{1}{3} x_1 \ge v$

$E(1, Y^*) \le v \quad \frac{1}{2} y_1 + 1(1 - y_1) = 1 - \frac{1}{2} y_1 \le v$

$E(1, Y^*) \le v \qquad \frac{1}{2} y_1 + 1(1-y_1) = 1 - \frac{1}{2} y_1 \le v$

$E(2, Y^*) \le v \qquad 1\cdot y_1 + \frac{2}{3}(1-y_1) = \frac{2}{3} + \frac{1}{3} y_1 \le v$

(same graph as for x_1 vs v)



intersection gives $x_1 = \frac{2}{5}$ so $x_2\ \frac{3}{5}$ $\quad v = \frac{4}{5}$

and $\quad y_1 = \frac{2}{5} \quad y_2 = \frac{3}{5}$

$X^* = \left(\frac{2}{5}\ \frac{3}{5}\right) = Y^*$ is saddle point

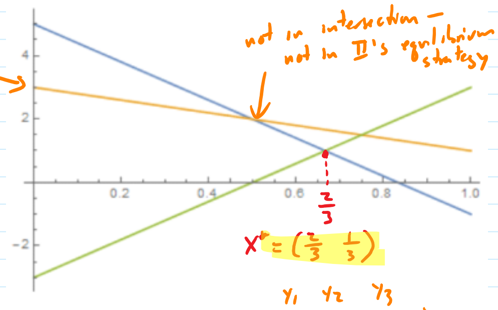
$$A = \begin{pmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \end{pmatrix}$$

$(x_1, 1-x_1)$

$E(x,1) = 1\cdot x_1 + (1-x_1)\cdot 3 = 3 - 2x_1 \ge v$

$E(x,2) = -1\cdot x_1 + (1-x_1)\cdot 5 = 5 - 6x_1 \ge v$

$E(x,3) = 3\cdot x_1 + (1-x_1)\cdot -3 = -3 + 6x_1 \ge v$

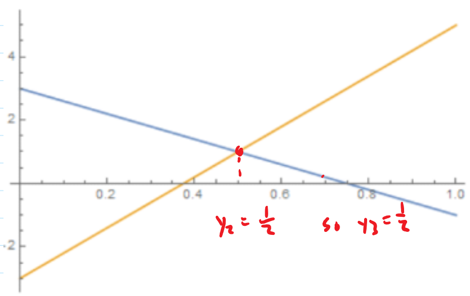


$X^* = \left(\frac{2}{3}\ \frac{1}{3}\right)$

$E(1,Y) = -1\cdot y_2 + 3(1-y_2) = 3 - 4y_2 \le v$

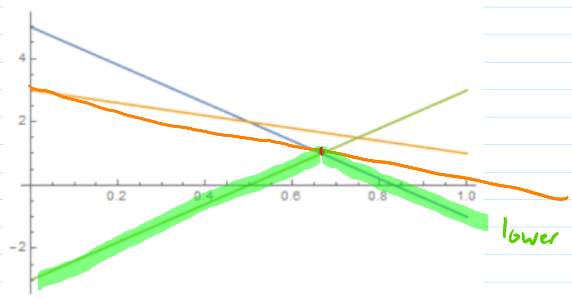
$E(2,Y) = 5\cdot y_2 + -3\cdot(1-y_2) = -3 + 8y_2 \le v$

$$\begin{array}{cccc} & y_1 & y_2 & y_3 \\ Y^* = (& 0 & y_2 & 1-y_2 \ ) \end{array}$$



$Y^* = \left(0\ \frac{1}{2}\ \frac{1}{2}\right)$

$$A = \begin{pmatrix} 0 & -1 & 3 \\ 3 & 5 & -3 \end{pmatrix}$$



Linear Programming

	F	C	S
F	0.30	0.25	0.20
C	0.26	0.33	0.28
S	0.28	0.30	0.33

Maximize v = minimize $\frac{1}{v}$ subject to

$E(X,1) = 0.30x_1 + 0.26x_2 + 0.28x_3 \geq v$

$E(X,2) = 0.25x_1 + 0.33x_2 + 0.30x_3 \geq v$

$E(X,3) = 0.20x_1 + 0.28x_2 + 0.33x_3 \geq v$

$x_1 + x_2 + x_3 = 1$

$x_1, x_2, x_3 \geq 0$

$\frac{x_1}{v} = p_1 \quad \frac{x_2}{v} = p_2 \ldots$

divide by v and $\cdot -1$

$0.20 \leq v \leq 0.33$

min payoff / max payoff

positive, so :) (if not, add C to all payoffs to make positive) and subtract out after getting soln for value of original game

a_1 / b_ub

$-0.30p_1 - 0.26p_2 - 0.28p_3 \leq -1$

$-0.25p_1 - 0.33p_2 - 0.30p_3 \leq -1$

$-0.20p_1 - 0.28p_2 - 0.33p_3 \leq -1$

for Java, convert to =

$-0.30p_1 - 0.26p_2 - 0.28p_3 + 1s_1 + 0s_2 + 0s_3 = -1$

$-0.25p_1 - 0.33p_2 - 0.33p_3 + 0s_1 + 1s_2 + 0s_3 = -1$

$\vdots$

$0 \leq s_i \leq 1$

$\vdots$

bounds

$0 \leq p_1 \leq 5$

$0 \leq p_2 \leq 5$

$0 \leq p_3 \leq 5$

$p_i = \frac{x_i}{v}$

$x_i = 1$

$v \geq .2$

$\frac{1}{v} \leq 5$

minimize $\underbrace{p_1 + p_2 + p_3}_{c} = \frac{x_1}{v} + \frac{x_2}{v} + \frac{x_3}{v} = \frac{x_1+x_2+x_3}{v} \approx \frac{1}{v}$

lin prog returns p_i's and $\frac{1}{v}$; convert back to x_i's and v

II

minimize v subject to

$E(1,Y) = 0.30y_1 + 0.25y_2 + 0.20y_3 \leq v$

$E(2,Y) = 0.26y_1 + 0.33y_2 + 0.28 \leq v$

$E(3,Y) = 0.28y_1 + 0.30y_2 + 0.33 \leq v$

$y_1 + y_2 + y_3 = 1$

$y_1, y_2, y_3 \geq 0$

$\Downarrow$ divide by v, set $q_i = \frac{y_i}{v}$ (don't need to multiply by -1 b/c already $\leq$)

a_2

$0.30q_1 + 0.25q_2 + 0.20q_3 \leq 1$

$$a_2 \quad \begin{aligned} 0.30\,q_1 + 0.25\,q_2 + 0.20\,q_3 &\le 1 \\ 0.26\,q_1 + 0.33\,q_2 + 0.28\,q_3 &\le 1 \\ 0.28\,q_1 + 0.30\,q_2 + 0.33\,q_3 &\le 1 \end{aligned} \quad b_{ub}$$

$$\begin{aligned} 0 \le\ & q_1 \le 5 \\ 0 \le\ & q_2 \le 5 \\ 0 \le\ & q_3 \le 5 \end{aligned} \quad \text{bounds}$$

minimize $\underbrace{-q_1 + -q_2 + -q_3}_{c} = -\frac{1}{v}$ (same as max $\frac{1}{v}$, same as min v)

$$E_I(X, Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} \underline{x_i \cdot y_j} \cdot a_{ij}$$

nonlinear (and appear in objective fxn in system to optimize)

$$E_{II}(X, Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} \underline{x_i \cdot y_j} \cdot b_{ij}$$

non-linear programming

NFL Strategy

field position, down + distance, time remaining

Let $E[pos] = P(\text{offense wins}) = P(\text{score TD on current possession})$

simplifying assumption

Then $E[pos] = \text{value} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{21} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

where a_{ij} = value of running offensive play i against defensive play j

$= \sum_{\text{outcome } o} P(o) \cdot E[next(pos, o)]$

position that results from outcome o in position pos

(yards gained, time elapsed, turnover)

So need dynamic programming + linear programming

already done! E [pos] in pickle file / csv file for all positions pos

1) populate matrix, adjust guarantee positive value
2) set up linear programs
3) display results (prob dist for offense + defense + value of game)