1 Motivation and Background

Define an $\epsilon$-expander to be a $d$-regular graph s.t. $|d - \lambda_i| \leq \epsilon d$ for $i \geq 2$ where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are eigenvalues of the Laplacian of our graph. In an $\epsilon$-expander, all the non-zero eigenvalues ($\lambda_2, \cdots, \lambda_n$) are close to $d$.

For a complete graph $K_n$, all non-zero eigenvalues of the Laplacian are $n$. Thus, the eigenvalues of an $\epsilon$-expander $G$ with degree $d$ are close to, or approximate the complete graph $\frac{d}{n}K_n$. Thus $\epsilon$-expanders can be regarded as sparsifiers. We can then do computations on expanders that are much more computationally appealing than those on the complete graph.

We are interested in $d$-regular expanders for large graphs, those with a high number of vertices. Intuitively, as the diameter of the graph increases with the number of vertices, the expander looks less like the complete graph since it has fixed degree $d$. Thus, there should be a way to formally quantify how poor an approximation of a complete graph an expander can be for large $n$. One way this can be done is by looking at an upper bound for $\lambda_2$ that is less than $d$ and a lower bound for $\lambda_n$ that is greater than $d$.

A theorem of A. Nilli states that:

**Theorem 1.** Let $G$ be a $d$-regular graph containing two edges that are at least $2k+2$ apart.
Then:
\[
\lambda_2 \leq d - 2\sqrt{d-1} + \frac{2\sqrt{d-1}-1}{k+1}
\]

Let us note a couple of things. As \(n\) gets very large, the hypothesis for this theorem is guaranteed to be satisfied because \(d\) is fixed. Furthermore, as \(n\) gets large, \(k\) gets large as well and we can treat the third term on the right hand side as an error term.

The proof for this theorem constructs a test vector that starts at \(\pm 1\) on either end of the diameter of the graph and decreases as we move toward the center of the graph. Applying this test vector to the Rayleigh quotient of the Laplacian gives us this bound.

2 Thesis

The goal of this senior thesis is to continue on this idea and provide a lower bound for the spectral gap, thus establishing a bound on how good \(d\)-regular expanders can be for large graphs. This can be done by constructing a lower bound on \(\lambda_n\). However, the proof technique employed on second smallest eigenvalue does not work with a direct application to \(\lambda_n\). There can be examples constructed such that the lower bound on the Rayleigh quotient is worse than the trivial bound of \(d\).

However, these examples are precisely the ones where we can further lower the upper bound on \(\lambda_2\). Thus, this suggests looking at the spectral gap directly or \(\lambda_n - \lambda_2\). The goal is to prove something like \(\lambda_n - \lambda_2 \geq 4\sqrt{d-1}\).

If this is done, one can then look at the similar problem for irregular graphs where the average degree is \(d\) and then look at weighted graphs. By this we mean considering graphs that \(1+\epsilon\)-approximate a complete graph with average weighted degree \(d\). As noted in [Bat14], such a graph has high edge conductance and rapid mixing. Most importantly, such a graph also has the expander mixing property. Thus, such a graph is very much like an expander and it
is conjectured that a similar bound on the spectral gap can be established.

3 Deliverables

It is believed that the conjectures made above are true but it is unknown if they actually are. Therefore, promising a proof as a deliverable seems optimistic. The deliverables for this thesis thus are attempts (and hopefully successes) of the above proof and any code used to test examples/counter-examples for the above conjectures.

These attempts/successes will be written up in paper-form with necessary background and motivation. It will be similar to the above but in more explicit detail.

4 Bibliography
